Complex and Non-Complex Phase Structures in Models of Spin Glasses and Information Processing

Hidetoshi Nishimori

Department of Physics, Tokyo Institute of Technology, Oh-okayama, Meguro-ku, Tokyo 152-8551, Japan

Abstract

The gauge theory of spin glasses and statistical-mechanical formulation of error-correcting codes are reviewed with an emphasis on their similarities. For the gauge theory, we explain the functional identities on dynamical autocorrelation functions and on the distribution functions of order parameters. These functional identities restrict the possibilities of slow dynamics and complex structure of the phase space. An inequality for error-correcting codes is shown to be interpreted to indicate non-monotonicity of spin orientation as a function of the temperature in spin glasses.

Key words: spin glass, gauge theory, complex phase space, slow dynamics, error-correcting codes

1 Introduction

The mean-field theory of spin glasses is well established [1]. There exists a low-temperature spin glass phase with complex structure of the phase space with such characteristics as replica symmetry breaking, many valleys and slow dynamics. In contrast, investigations of realistic finite-dimensional systems have been hampered by difficulties of analytic treatment, and numerical methods have been the predominant tool of research. An interesting exception is the gauge theory which exploits gauge symmetry of the system to derive many exact/rigorous results such as the exact expression of the internal energy, a bound on the specific heat and various identities for correlation functions [1,2].

It may be a little bit surprising that this gauge theory is closely related with some problems of information processing such as error-correcting codes and image restoration. Suppose that one transmits digital information (a bit sequence) through a noisy channel. An important goal of error-correcting codes

is to devise an efficient way to encode and decode the information in order to remove as much noise as possible from the output of the channel. This problem can be rewritten as an Ising spin glass [3]: A bit sequence is naturally expressed as an Ising spin configuration, and noise corresponds to randomness in exchange interactions. Then the statistical inference of the original information, given the noisy output of the channel, is found to be equivalent to the statistical mechanics of an Ising spin glass.

A similar analogy exists between image restoration, in which one tries to remove noise from a given digital image, and the statistical mechanics of a spin system in random fields. It is the purpose of this contribution to review the gauge theory of spin glasses and information processing problems, error-correcting codes in particular, with emphasis on the similarities between these superficially quite different fields [1].

The gauge theory of spin glasses and error-correcting codes are reviewed in sections 2 and 3, respectively, with a few new results and viewpoints. Summary is given in section 4.

2 Gauge theory of spin glasses

The gauge theory of spin glasses is a powerful theoretical framework to derive a number of exact/rigorous results on models of spin glasses using gauge symmetry of the system. It does not answer directly the question of the existence or absence of the spin glass phase in finite dimensions. Nevertheless this theory leads to various results which set strong constraints on the possibilities of the existence region of the spin glass phase in the phase diagram and other non-trivial conclusions [1,2,4].

2.1 System and its symmetry

Let us consider the $\pm J$ Ising model with the Hamiltonian

$$H = -\sum_{\langle ij \rangle} J_{ij} S_i S_j, \tag{1}$$

where J_{ij}, a quenched random variable, is either J (with probability p) or $-J$ (with probability $1-p$). The sum over $\langle ij \rangle$ runs over appropriate pairs of sites, such as (but not restricted to) nearest neighbours on a regular lattice. This

probability distribution can be written in a compact form as

$$P(J_{ij}) = \frac{e^{K_p \tau_{ij}}}{2\cosh K_p}, \tag{2}$$

where $\tau_{ij}(=\pm 1)$ is the sign of J_{ij}. It is easy to check that (2) is equal to p if $\tau_{ij} = 1$ and is $1-p$ when $\tau_{ij} = -1$ if we define the parameter K_p by $e^{-2K_p} = (1-p)/p$.

The Hamiltonian (1) is invariant under the gauge transformation

$$S_i \to S_i\sigma_i, \quad J_{ij} \to J_{ij}\sigma_i\sigma_j, \tag{3}$$

where $\{\sigma_i\}$ is a set of Ising variables fixed arbitrarily at each site.

2.2 Exact energy

It is possible to derive the exact expression of the internal energy, averaged over the probability (2), under a simple condition between the temperature and the probability p. The definition of the internal energy is

$$[E] = \sum_{\{\tau_{ij}\}} \prod_{\langle ij\rangle} P(J_{ij}) \cdot \frac{\sum_{\{S_i\}} He^{-\beta H}}{\sum_{\{S_i\}} e^{-\beta H}}, \tag{4}$$

where the square brackets $[\cdots]$ denote the configurational average. The gauge transformation (3) just redefines the running variables $\{\tau_{ij}\}$ and $\{S_i\}$ and therefore does not affect the value of the internal energy. Thus (4) is rewritten as

$$[E] = \sum_{\tau} \frac{e^{K_p \sum \tau_{ij}\sigma_i\sigma_j}}{(2\cosh K_p)^{N_B}} \cdot \frac{\sum_S He^{K\sum \tau_{ij}S_iS_j}}{\sum_S e^{K\sum \tau_{ij}S_iS_j}}, \tag{5}$$

where N_B is the number of bonds in the system and $K = \beta J$. Since the value of (5) is independent of the choice of $\{\sigma_i\}$, we may sum the right hand side of (5) over all possible values of $\{\sigma_i\}$ and divide the result by 2^N without changing the value of $[E]$:

$$[E] = \frac{1}{(2\cosh K_p)^{N_B} 2^N} \sum_{\tau}\sum_{\sigma} e^{K_p \sum \tau_{ij}\sigma_i\sigma_j} \cdot \frac{\sum_S He^{K\sum \tau_{ij}S_iS_j}}{\sum_S e^{K\sum \tau_{ij}S_iS_j}}. \tag{6}$$

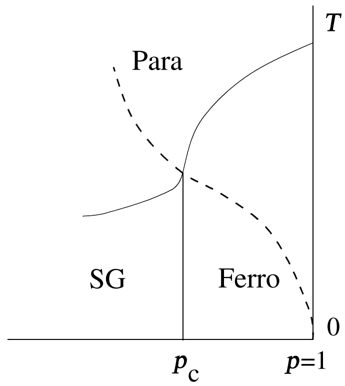

Fig. 1. Phase diagram of the $\pm J$ model and the Nishimori line (dashed).

It is clear here that the denominator on the right hand side cancels with the sum over σ if $K = K_p$, which makes it possible to evaluate the energy explicitly:

$$\begin{aligned}[E] &= \frac{1}{(2\cosh K_p)^{N_B} 2^N} \sum_\tau \sum_S H e^{K \sum \tau_{ij} S_i S_j} \\ &= \frac{1}{(2\cosh K_p)^{N_B} 2^N} \left(-\frac{\partial}{\partial \beta}\right) \sum_\tau \sum_S e^{K \sum \tau_{ij} S_i S_j} \\ &= \frac{1}{(2\cosh K_p)^{N_B} 2^N} \left(-\frac{\partial}{\partial \beta}\right) \sum_S (2\cosh K)^{N_B} \\ &= -N_B J \tanh K. \end{aligned} \tag{7}$$

The only condition for the above manipulations to be valid is the relation $K = K_p$ to relate the temperature with the probability, defining a line in the phase diagram, the Nishimori line shown dashed in Fig. 1. Note that the lattice structure, range of interaction, or the spatial dimension are all arbitrary. So the present exact solution (7) is very generic.

Similar arguments lead to an upper bound on the specific heat and various identities for static correlation functions under the condition $K = K_p$.

2.3 Absence of slow dynamics

An interesting relation can be established for dynamical correlation functions using the gauge theory [5,6]:

$$\left[\langle S_i(t_w) S_i(t+t_w)\rangle^{\mathrm{F}}_{K_p}\right] = \left[\langle S_i(0) S_i(t)\rangle^{\mathrm{eq}}_{K_p}\right]. \tag{8}$$

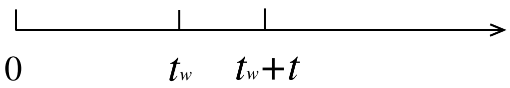

Fig. 2. Two measurement times of the dynamical correlation function on the right hand side of (8).

The right hand side is the equilibrium autocorrelation function at temperature $K = K_p$. The left hand side is a non-equilibrium correlation function starting from the completely ferromagnetic state at time 0. The first measurement is performed after waiting time t_w and the second at $t_w + t$ (Fig. 2). If the systems shows anomalously slow dynamics, the non-equilibrium correlation function, the left hand side, will depend on the measurement starting time (the waiting time t_w) as it actually happens in the spin glass phase of the mean-field model. Otherwise, the system relaxes quickly to equilibrium, implying time-translation invariance, or equivalently, independence of the left hand side of t_w. The identity (8) proves that the latter possibility is the case since the right hand side does not depend on t_w. Therefore, any spin glass system does not show slow dynamics on the line $K = K_p$. The proof of (8) requires some background of the kinetic Ising model and we refer the reader to the original papers.

Note that the above argument is not a proof of the absence of a spin glass phase in general. What has been shown is that a phase with slow dynamics should exist, if it does at all, away from the Nishimori line.

2.4 Distribution functions of order parameters

An interesting functional identity $P_m(x) = P_q(x)$ can be proved under the same condition $K = K_p$ [7]. Here, $P_m(x)$ is the distribution function of magnetization and $P_q(x)$ is the distribution of spin glass order parameter,

$$P_m(x) = \left[\frac{\sum_S \delta\left(x - \frac{1}{N}\sum_i S_i\right) e^{-\beta H}}{\sum_S e^{-\beta H}} \right] \tag{9}$$

and

$$P_q(x) = \left[\frac{\sum_S \sum_\sigma \delta\left(x - \frac{1}{N}\sum_i S_i\sigma_i\right) e^{-\beta H(S) - \beta H(\sigma)}}{\sum_S \sum_\sigma e^{-\beta H(S) - \beta H(\sigma)}} \right]. \tag{10}$$

The proof of the identity $P_m(x) = P_q(x)$ is relatively straightforward if we apply the gauge transformation to the expression (9) and use the condition $K = K_p$. After the gauge transformation, (9) is expressed as

$$P_m(x) = \frac{1}{(2\cosh K_p)^{N_B} 2^N} \sum_\tau \sum_\sigma e^{K_p \sum \tau_{ij}\sigma_i\sigma_j}$$
$$\cdot \frac{\sum_\sigma \sum_S \delta\left(x - \frac{1}{N}\sum_i S_i\sigma_i\right) e^{K_p \sum \tau_{ij}\sigma_i\sigma_j} e^{K \sum \tau_{ij} S_i S_j}}{\sum_\sigma e^{K_p \sum \tau_{ij}\sigma_i\sigma_j} \sum_S e^{K \sum \tau_{ij} S_i S_j}}. \tag{11}$$

The last expression is equivalent to (10) if $K = K_p$.

Since the distribution function of magnetization $P_m(x)$ is always composed of at most two simple delta functions located at $\pm m$, the identity $P_m(x) = P_q(x)$ implies that the distribution of spin glass order parameter $P_q(x)$ also has the same simple structure. This excludes the possibility that a complex structure of the phase space, which should be reflected in a non-trivial functional form of $P_q(x)$, exists on the line $K = K_p$. Thus a spin glass (or mixed ferromagnetic) phase of the mean-field type, if it exists, should lie away from the line in the phase diagram. This observation reinforces the conclusion of the previous subsection on dynamics that there is no mean-field type spin glass phase on the Nishimori line.

3 Error-correcting codes

The arguments in the previous section on the gauge theory of spin glasses are in close formal similarity to some problems of information processing. We demonstrate it with error-correcting codes as a typical example [1].

3.1 Transmission of information

The goal of the theory of error-correcting codes is to find efficient methods to infer the original information (bit sequence) from the output of a noisy transmission channel. For this purpose it is known to be necessary to introduce redundancy before sending the information through the channel, a process called encoding (or coding), see Fig. 3. At the receiving end of the channel, one decodes the signal to retrieve the original information making full use of the redundancy. Shannon's channel coding theorem sets a lower limit on the redundancy in order to retrieve the original information without errors, given the noise strength of the channel.

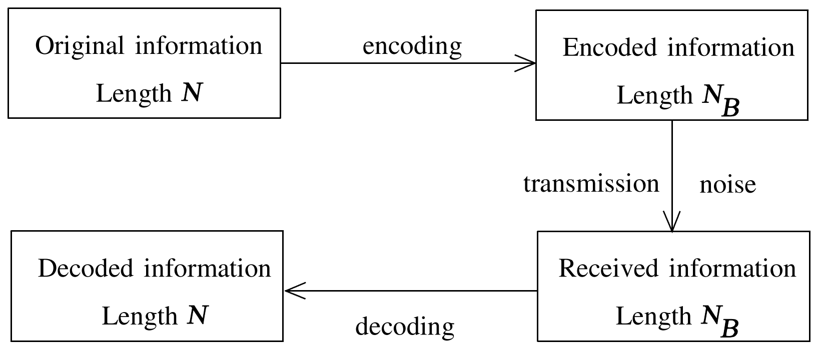

Fig. 3. The original information of N bits is encoded to a binary sequence of $N_B(> N)$ bits which is then transmitted through the noisy channel. The output of the channel is decoded to infer the original information.

3.2 Ising representation

It is convenient to formulate the problem in terms of Ising spins as natural correspondences exist between a bit and an Ising spin (0 and 1 corresponds to $S_i = (-1)^0 = 1$ and $S_i = (-1)^1 = -1$, respectively) and between the mod-two addition and the multiplication (e.g., 1+1=0 is equivalent to $(-1)^1 \times (-1)^1 = (-1)^0 = 1$) [3]. Suppose that the original information one wishes to send is an Ising spin configuration $\{\xi_i\}$. Redundancy can be introduced by generating a set of many-body Mattis-type interactions [8] from the spin configuration, $J^0_{ijk\cdots} = \xi_i\xi_j\xi_k\cdots$. Clearly the number of interactions N_B is larger than the number of Ising spins N, implying redundancy. A specific choice of the set of sites to form interactions $\{ijk\cdots\}$ corresponds to a specific code. A search for a good code is one of the main targets of coding theory, but we do not review this problem here and continue the argument without specifying the set $\{ijk\cdots\}$.

The encoded information $\boldsymbol{J}^0 \equiv \{J^0_{ijk\cdots}\}$ is transmitted through the noisy channel. Let us focus our attention on a binary symmetric channel in which the input $J^0_{ijk\cdots}$ is flipped to the opposite sign $J_{ijk\cdots} = -J^0_{ijk\cdots}$ with error probability p. The task is to infer (decode) the original information $\{\xi_i\}$ from the noisy output of the channel $\boldsymbol{J} \equiv \{J_{ijk\cdots}\}$. Note that frustration exists in the set of interactions $\boldsymbol{J}$ although $\boldsymbol{J}^0$ (which was generated by the Mattis rule) is not frustrated. Finite-temperature statistical mechanics of the Ising spin glass with interactions $\boldsymbol{J}$ serves as a useful theoretical platform to decode the message as shown in the following [3,9–12].

3.3 Bayesian formulation

The first step is to express the noise characteristics using the conditional probability as

$$P(J_{ijk\cdots}|J^0_{ijk\cdots}) = P(J_{ijk\cdots}|\xi_i\xi_j\xi_k\cdots) = \frac{e^{\beta_p J_{ijk\cdots}\xi_i\xi_j\xi_k\cdots}}{2\cosh\beta_p} \tag{12}$$

which is equal to p if $J_{ijk\cdots} = -J^0_{ijk\cdots}$ and $1-p$ otherwise if we define β_p as

$$e^{2\beta_p} = \frac{1-p}{p}. \tag{13}$$

Note that β_p is essentially equivalent to K_p appeared in the previous section, the only difference being that p and $1-p$ are exchanged. Assuming a memoryless channel, in which each bit is affected by noise independently of other bits, the conditional probability of the whole spin configuration is

$$P(\boldsymbol{J}|\boldsymbol{\xi}) = \prod_{\{ijk\cdots\}} P(J_{ijk\cdots}|\xi_i\xi_j\xi_k\cdots) = \frac{\exp(\beta_p\sum_{\{ijk\cdots\}} J_{ijk\cdots}\xi_i\xi_j\xi_k\cdots)}{(2\cosh\beta_p)^{N_B}}, \tag{14}$$

where N_B is the size of the set $\boldsymbol{J}$.

Equation (14) represents the conditional probability of the output $\boldsymbol{J}$ given the input $\boldsymbol{\xi}$. To infer the original information, we need the conditional probability of the input $\boldsymbol{\xi}$ given the output $\boldsymbol{J}$. The Bayes formula is useful for this purpose:

$$P(\boldsymbol{\sigma}|\boldsymbol{J}) = \frac{P(\boldsymbol{J}|\boldsymbol{\sigma})P(\boldsymbol{\sigma})}{\sum_\sigma P(\boldsymbol{J}|\boldsymbol{\sigma})P(\boldsymbol{\sigma})}, \tag{15}$$

where $\boldsymbol{\sigma}$ is a set of dynamical variables used to infer $\boldsymbol{\xi}$.

We need the explicit form of the distribution function of the original information $P(\boldsymbol{\sigma})$, called the prior, to use the Bayes formula (15) to infer $\boldsymbol{\sigma}$ given $\boldsymbol{J}$. It can reasonably assumed that $P(\boldsymbol{\sigma})$ is a constant, a uniform prior, because one often compresses information before encoding, which usually generates a very uniform distribution of 0s and 1s. Then the $\boldsymbol{\sigma}$-dependence of the right hand side comes only from $P(\boldsymbol{J}|\boldsymbol{\sigma})$, and thus we find

$$P(\boldsymbol{\sigma}|\boldsymbol{J}) \propto P(\boldsymbol{J}|\boldsymbol{\sigma}) \propto \exp(\beta_p \sum_{\{ijk\cdots\}} J_{ijk\cdots}\sigma_i\sigma_j\sigma_k\cdots). \tag{16}$$

The right hand side is the Boltzmann factor of the Ising spin glass with many body interactions which are quenched since $\boldsymbol{J}$ is given and fixed. Therefore the problem has been reduced to the statistical mechanics of the Ising spin glass with the effective coupling β_p.

3.4 Statistical inference

There are two typical methods to infer the original information (spin configuration) from the conditional probability (16) called the posterior. One is the MAP (maximum a posteriori probability) method in which one chooses the configuration $\boldsymbol{\sigma}$ that maximizes $P(\boldsymbol{\sigma}|\boldsymbol{J})$. This is equivalent to the ground-state search of the Ising spin glass

$$H = -\sum_{\{ijk\cdots\}} J_{ijk\cdots}\sigma_i\sigma_j\sigma_k\cdots \tag{17}$$

according to (16). Another is the MPM (maximizer of posterior marginals) method; one first marginalizes the posterior with respect to σ_i

$$P(\sigma_i|\boldsymbol{J}) = \sum_{\sigma_j(j\neq i)} P(\boldsymbol{\sigma}|\boldsymbol{J}) \tag{18}$$

and chooses σ_i to maximize this marginal probability. If we write the inferred value as $\hat{\xi}_i$, we have $\hat{\xi}_i = \arg\max_{\sigma_i} P(\sigma_i|\boldsymbol{J})$. Such a process is carried out at each i. The MPM is equivalent to looking at the sign of the local magnetization $\hat{\xi}_i = \mathrm{sgn}\langle\sigma_i\rangle$, where the thermal average is taken using the Boltzmann factor (16) of the Ising spin system with effective coupling β_p. The MAP may be regarded as the same method $\hat{\xi}_i = \mathrm{sgn}\langle\sigma_i\rangle$, the only difference being that the thermal average is now evaluated at the ground state. Thus these two methods may be called the ground-state decoding (MAP) and finite temperature decoding (MPM), respectively.

An important measure of decoding performance is the overlap of the decoded information (spin configuration) $\{\hat{\xi}_i\}$ and the original true information $\{\xi_i\}$,

$$M = \frac{1}{N}\left[\sum_i \xi_i\hat{\xi}_i\right], \tag{19}$$

where the square brackets denote the configurational average with respect to the distribution of noise (or equivalently, quenched randomness in interactions). Perfect decoding result gives $M = 1$ whereas a random result yields $M = 0$. A larger M represents a better result.

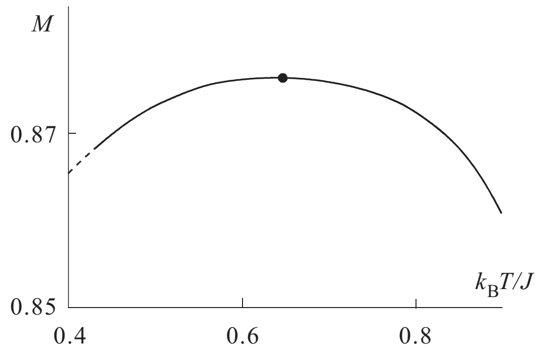

Fig. 4. Overlap as a function of the temperature

It is useful to replace the parameter β_p in (16) with a general β because, first, the MAP ($\beta \to \infty$) and the MPM ($\beta = \beta_p$) can be treated within the same framework and, second, one sometimes does not know the correct error rate p (and its function β_p) in applying the MPM. The overlap is then a function of β and p. It turns out that the overlap $M(\beta, p)$ as a function of β is not monotonic but reaches the maximum at β_p (Fig. 4) [10]:

$$M(\beta, p) \leq M(\beta_p, p). \tag{20}$$

This non-monotonicity is not surprising intuitively because the overlap has been defined as a measure of average bitwise performance (19) and the MPM has been designed to maximize the bitwise posterior (18). The proof of the inequality (20) will be given in the next subsection.

3.5 Interpretation as the Ising spin glass

The inequality (20) has the following interesting interpretation in the context of spin glasses. Let us consider the problem in terms of the ferromagnetic gauge, $\xi_i = 1$ ($\forall i$). Generality is not lost by this restriction because the prior has been assumed to be uniform, $P(\boldsymbol{\sigma})$ =const, and thus all the original configurations have the same statistical properties, leading to the same average decoding performance $M(\beta, p)$ for any choice of $\{\xi_i\}$.

In the ferromagnetic gauge the overlap $M(\beta, p)$ represents the average of the difference between the numbers of up spins and down spins:

$$M(\beta, p) = \frac{1}{N} \left[\sum_i \hat{\xi}_i \right] = [\mathrm{sgn}\langle \sigma_i \rangle] \,. \tag{21}$$

It should also be noticed that (16) is equivalent to the Boltzmann factor of the $\pm J$ Ising model with many body interactions only for the ferromagnetic

gauge, strictly speaking. The reason is that the interaction $J_{ijk\cdots}(=\xi_i\xi_j\xi_k\cdots)$ is equal to $+1$ with probability $1-p$ and to -1 with probability p only in the ferromagnetic gauge. Thus the effective inverse temperature β_p in (16) represents the condition that the system is on the Nishimori line $\beta_p=\beta$ in the ferromagnetic gauge (with p and $1-p$ exchanged from the notation of section 2).

To prove (20) in the ferromagnetic gauge, we first write (21) explicitly:

$$\left[\frac{\langle\sigma_i\rangle}{|\langle\sigma_i\rangle|}\right]=\frac{1}{(2\cosh\beta_p)^{N_B}}\sum_\tau e^{\beta_p\sum\tau_{ij}}\frac{\sum_\sigma\sigma_i e^{\beta\sum\tau_{ij}\sigma_i\sigma_j}}{|\sum_\sigma\sigma_i e^{\beta\sum\tau_{ij}\sigma_i\sigma_j}|}. \tag{22}$$

Then, by applying the gauge transformation (with gauge variable η_i) and evaluating the upper bound by taking the absolute value of the summand, we find

$$\begin{aligned}\left[\frac{\langle\sigma_i\rangle}{|\langle\sigma_i\rangle|}\right]&=\frac{1}{2^N(2\cosh\beta_p)^{N_B}}\sum_\eta\sum_\sigma e^{\beta_p\sum\tau_{ij}\eta_i\eta_j}\langle\eta_i\rangle_{\beta_p}\frac{\langle\sigma_i\rangle_\beta}{|\langle\sigma_i\rangle_\beta|}\\&\le\frac{1}{2^N(2\cosh\beta_p)^{N_B}}\sum_\eta\sum_\sigma e^{\beta_p\sum\tau_{ij}\eta_i\eta_j}|\langle\eta_i\rangle_{\beta_p}|.\end{aligned} \tag{23}$$

The last expression is easily seen to be equal to $[\mathrm{sgn}\langle\sigma_i\rangle_{\beta_p}]=M(\beta_p,p)$ if we apply the gauge transformation to $[\mathrm{sgn}\langle\sigma_i\rangle_{\beta_p}]$.

The inequality (20) is interpreted that the number of up spins, relative to the number of down spins, is a non-monotonic function of the temperature. The number of up spins reaches its maximum on the Nishimori line and then decreases as one further lowers the temperature (Fig. 4). In this sense, the spin state on the Nishimori line is more ordered than the states at any other temperature with the same p.

This is a highly non-trivial result. As one lowers the temperature from the paramagnetic phase, the number of up spins becomes to exceed that of down spins as soon as the system enters the ferromagnetic phase (point A in Fig. 5) if one imposes the boundary condition such that the overall inversion symmetry is broken for the system with two-body interactions. [1] The number of up spins reaches the maximum on the Nishimori line, where, in some sense, thermal fluctuations are balanced with geometrical frustration (point B). Then, as one further lowers the temperature, the spin states start to capture the detailed structure of bond configurations, which were masked by thermal fluctuations

[1] Note that the argument in the present section applies to the usual case of two-body interactions as well.

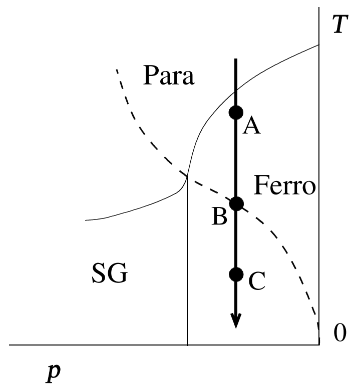

Fig. 5. The spin state is more ordered at point B than at A or C.

above the Nishimori line, and the number of up spins starts to decrease at lowering the temperatures (point C).

It should be remembered here that the magnetization $m(\beta, p) = [\langle \sigma_i \rangle_\beta]$, not the overlap $M(\beta, p) = [\mathrm{sgn}\langle \sigma_i \rangle_\beta]$, is of course expected to be a monotonic function of the temperature. Only the number of up spins $M(\beta, p)$, which ignores the spin-size reduction due to thermal fluctuations ($|\langle \sigma_i \rangle| < |\mathrm{sgn}\langle \sigma_i \rangle| = 1$), is non-monotonic. We may therefore conclude that the Nishimori line marks the crossover between the purely ferromagnetic region (where M increases as the temperature is decreased) and a randomness-dominated region (where M decreases). This observation is reinforced by the fact that the ferromagnetic order parameter m is equal to the spin glass order q on the Nishimori line as can be derived from the functional identity $P_m(x) = P_q(x)$ mentioned in section 2: We reasonably expect that the ferromagnetic order dominates $m > q$ above the Nishimori line and the opposite is true $m < q$ below.

Another interesting and useful aspect concerning the relation (or equivalence) between the Ising spin glass and error-correcting codes is the simplicity of phase space and the absence of slow dynamics on the Nishimori line discussed in section 2. Practical algorithms of decoding are generally implemented as iterative solutions of TAP-like equations for local magnetizations. Such iterations can be regarded as discrete-time dynamics of the Ising spin glass. And the iteration is often carried out on the Nishimori line to achieve the best performance in the sense of maximum overlap M. Then, if the system shows slow dynamics, the iteration may not converge to the desired result in a reasonable amount of time. The results in section 2 guarantee that this does not happen. [2]

[2] There still exists a problem of appropriate choice of the initial condition in practice [13].

4 Summary

Exact/rigorous results have been presented on static and dynamical properties of the Ising spin glass using the gauge theory. These results have been shown to restrict the possibilities of complex structure of the phase space and slow dynamics of the mean-field type. We have presented a Baysian formulation of error-correcting codes that has a very close formal similarity to the Ising spin glass. The best bitwise inference of the original information has been shown to be achieved by a finite-temperature decoding corresponding to the Nishimori line derived in the gauge theory of spin glasses. We have seen that an inequality on the overlap, which is a measure of decoding performance of error-correcting codes, has an interesting interpretation under the context of spin glasses that the average spin orientation is not a monotonic function of the temperature. The spin ordering, in the sense of the number of up spins (ignoring the spin reduction due to thermal fluctuations), takes its maximum value not at $T = 0$ but at a finite temperature corresponding to the Nishimori line. Thus the system of Ising spin glass is most ferromagnetically-ordered not in the ground state but at this finite temperature even within the ferromagnetic phase.

References

[1] H. Nishimori, *Statistical Physics of Spin Glasses and Information Processing : An Introduction*, Oxford Univ. Press (Oxford, 2001).

[2] H. Nishimori, Prog. Theor. Phys. **66**, 1169 (1981).

[3] N. Sourlas, Nature **339**, 693 (1989).

[4] Y. Ozeki and H. Nishimori, J. Phys. A **26**, 3399 (1993).

[5] Y. Ozeki, J. Phys. A **28**, 3645 (1995).

[6] Y. Ozeki, J. Phys. Cond. Matt. **9**, 11171 (1997).

[7] H. Nishimori and D. Sherrington, In *Disordered and Complex Systems*, Eds. P. Sollich, A. C. C. Coolen, L. P. Hughston and R. F. Streater, AIP Conference Proceedings 553 (AIP, Melville, New York, 2001).

[8] D. C. Mattis, Phys. Lett. A **56**, 421 (1976).

[9] P. Ruján, Phys. Rev. Lett. **70**, 2968 (1993).

[10] H. Nishimori, J. Phys. Soc. Jpn. **62**, 2973 (1993).

[11] N. Sourlas, Europhys. Lett. **25**, 159 (1994).

[12] Y. Iba, J. Phys. A **32**, 3875 (1999).

[13] Y. Kabashima, T. Murayama and D. Saad, Phys. Rev. Lett. **84**, 2030 (2000).